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Three Rods on a Ring and the Triangular Billiard

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We demonstrate the equivalence of two seemingly disparate dynamical systems. One consists of three hard rods sliding along a frictionless ring and making elastic collisions. The other consists of one ball moving on a frictionless triangular table with elastic rails. Several applications of this result are discussed.

KEY WORDS: Billiards; hard rods; impact phenomena; Tonks gas.

1. PROVING THE EQUIVALENCE

We show that the motion of three pointlike rods making elastic collisions along a frictionless ring of length L can be mapped onto that of one pointlike ball moving freely within a triangle and making elastic impacts with its legs. The masses of the rods are m_k and their velocities along the ring are v_k. When rods i and j collide, their relative velocity $v_i - v_j$ reverses, leaving the sum of their momenta $m_i v_i + m_j v_j$ unchanged. The conserved total momentum and energy are $P = \sum m_k v_k$ and $T = \frac{1}{2} \sum m_k v_k^2$, respectively, where sums here and henceforth extend over $k = 1, 2, 3$. With no loss of generality we assume $P = 0$, so that

$$\sum m_k v_k = 0 \tag{1.1}$$

Let x_k be the arclength between the other two rods via the route avoiding rod k. The positions of the rods can be expressed in terms of their fixed "center-of-mass" and these relative separations:

$$x_k \geqslant 0 \qquad \text{and} \qquad \sum x_k = L \tag{1.2}$$

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Using (1.1), we put T in terms of squares of the velocity differences, e.g., $\dot{x}_1 = v_2 - v_3$:

$$T = \frac{\Pi}{2M} \sum \frac{\dot{x}_k^2}{m_k}$$

with $M = \sum m_k$ and $\Pi = m_1 m_2 m_3$. We define

$$U_1 = (v_2 - v_3)\sqrt{\frac{m_2 m_3}{(m_2 + m_3)\,M}}, \qquad V_1 = v_1\sqrt{\frac{m_1}{m_2 + m_3}}, \qquad \text{\&c} \tag{1.3}$$

with &c indicating cyclic permutations. An impact in which rod k does not partake results in

$$U_k \to -U_k, \qquad V_k \to V_k, \qquad \text{where} \quad T = \tfrac{1}{2}M(U_k^2 + V_k^2) \tag{1.4}$$

The U_k, V_k pairs may be regarded as components of the same vector in different Cartesian coordinates:

$$\vec{W} = U_k \hat{e}_k + V_k \hat{f}_k, \qquad \text{with} \quad W^2 = \frac{\Pi}{M^2} \sum \frac{\dot{x}_k^2}{m_k} \tag{1.5}$$

$\vec{W}$ is identified as the velocity of a ball on the triangular table to be specified.

The three sets of basis vectors defined by (1.3) and (1.5) are related by rotations:

$$\begin{pmatrix} \hat{e}_2 \\ \hat{f}_2 \end{pmatrix} = -\begin{pmatrix} \cos\theta_3 & \sin\theta_3 \\ -\sin\theta_3 & \cos\theta_3 \end{pmatrix}\begin{pmatrix} \hat{e}_1 \\ \hat{f}_1 \end{pmatrix}, \qquad \text{\&c} \tag{1.6}$$

where

$$\cos\theta_3 = \sqrt{\frac{m_1 m_2}{(m_3 + m_2)(m_3 + m_1)}}, \qquad \sin\theta_3 = \sqrt{\frac{m_3 M}{(m_3 + m_2)(m_3 + m_1)}}, \qquad \text{\&c}$$

or equivalently:

$$m_k \cot\theta_k = \sqrt{\Pi/M} = M \cot\theta_1 \cot\theta_2 \cot\theta_3 \tag{1.7}$$

The θ_k lie in the first quadrant. The product of the three matrices defined by (1.6) and (1.7) reveals that $\sum \theta_k = \pi$.

The equivalent billiard table is an acute triangle with interior angles θ_k, legs l_k parallel to $\hat{f}_k$, and altitudes a_k parallel to $\hat{e}_k$. An interior point is given by its trilinear coordinates, the distances d_k from each leg:

$$d_k \geqslant 0 \qquad \text{and} \qquad \sum d_k l_k = l_1 a_1 = l_2 a_2 = l_3 a_3$$

The mapping between rod spacings and points in the triangle preserving (1.2) is

$$d_k l = a_k x_k \qquad \text{for} \quad k = 1, 2, 3 \tag{1.8}$$

Recognizing that $\dot{d}_k = U_k$, we find from (1.3) and (1.8)

$$d_1 = x_1 \sqrt{\frac{m_2 m_3}{(m_2 + m_3) M}}, \qquad \&c \tag{1.9}$$

Eliminating d_k from (1.8) and (1.9), we find the altitudes and legs of the triangle:

$$a_1 = L \sqrt{\frac{m_2 m_3}{(m_2 + m_3)\, M}}, \qquad l_1 = L \sqrt{\frac{m_1 (m_2 + m_3)}{M^2}}, \qquad \&c \tag{1.10}$$

Between impacts, the motion of the rods on the ring corresponds to uniform motion of the ball on the table. A rod–rod impact corresponds to the ball striking a leg of the triangle.[3] According to (1.4), the component of $\vec{W}$ perpendicular to the leg reverses and its parallel component is unchanged—precisely the result of a ball–rail impact. Q.E.D.

Equation (1.7) says that $\cot \theta_k \to 0$ as m_k tends to infinity with the other masses kept fixed. The triangular table becomes right rather than acute. We regain the well-known equivalence between the right-triangular billiard and the motion of two hard rods on an elastically bounded line segment.[1] Having found rods-on-a-ring motion equivalent to billiards on acute or right triangles, we ask whether obtuse triangles can play a role. Indeed they can, for the somewhat contrived case wherein rods 1 and 3 have *negative* masses $-m_1$ and $-m_3$, with $m_k > 0$ and $M = m_2 - m_1 - m_3 > 0$. As before, colliding rods reverse their relative velocity. In the center-of-mass system, $m_1 v_1 + m_3 v_3 = m_2 v_2$ and the energy is a *negative*-definite linear form in $\dot{x}_k^2$. Proceeding as above, we find the interior angles of the triangle: $\tan \theta_k = (-1)^{k+1} m_k \sqrt{M/\Pi}$. The motion of these rods maps onto that of a ball on a triangular table with $\theta_2 > \pi/2$.

[3] The ball striking a vertex of the triangle corresponds to a corner shot in the billiard and a three-rod impact on the ring. The result of such a collision is not always well defined.

2. USING THE EQUIVALENCE

Much of what is known about billiards on triangular tables[2,3] is directly applicable to the mechanical system of three elastic rods on a ring. Here are some examples:

(1) Any acute triangular table admits orbits of period six. Three rods on a ring with any positive masses display analogous periodic motions. They are realized for any initial positions of the rods if their initial relative velocities satisfy $m_2(m_1+m_3)\dot{x}_1+m_1(m_2+m_3)\dot{x}_3=0$, or any cyclic permutation thereof. The minimal period is six, unless two balls collide when the third is at a specific position on the ring, e.g., if $x_2=0$ when $m_3x_1=m_1x_3$. This special case corresponds to the pedal 3-orbit on an acute triangle, just as any billiard orbit with odd period n is a limiting case of orbits with period $2n$.[3,4]

(2) With the exceptions of 2, 8, 12, and 20, billiard orbits on the equilateral triangle can have any even period.[5] Orbits with even periods correspond to periodic motions of three identical rods with arbitrary initial positions along the ring. (Compare this result and that of the previous paragraph with Corollary 6 of ref. 3.) If the angles of the equivalent table are rational multiples of π, powerful billiard theorems[6] apply to the rod problem. However, rod masses corresponding to these rational triangles have no apparent physical significance.

(3) The following remark paraphrases and generalizes Corollary 1 of ref. 3 and follows from the work of Kerckhoff *et al.*[7]: The mechanical system of three elastic rods on a ring is typically ergodic.

(4) All nonperiodic orbits on any polygonal table come arbitrarily close to at least one vertex[4] (generalizing results of ref. 8). Thus, three rods on a ring in a nonperiodic orbit must come arbitrarily close to a triple collision.

(5) A generalization of our procedure maps the motion of $N+1$ rods with any masses onto that of one ball in an elastically bounded N-dimensional simplex, thus offering an alternative picture of the multicomponent Tonks gas.[9]

Conversely, rods moving on a ring can shed light on billiards. Let $\langle D\rangle$ be the mean distance between ball–rail impacts along a billiard trajectory. For the equilateral triangle of side l, the equivalent rod problem makes it obvious that $\langle D\rangle$ depends on the initial direction of motion but not the initial position,[4] and that $\langle D\rangle = l\sqrt{3}/(4\cos\phi)$, where ϕ is the

[4] This result is known to apply to billiard trajectories in almost every direction on any rational polygon.

smallest of the angles between the velocity of the ball and the normals to the legs of the triangle ($0 \leqslant \phi \leqslant \pi/6$). A well-known theorem of Birkhoff implies that the extrema of $\langle D \rangle$ on any convex table correspond to periodic orbits. We obtain an orbit of period six for $\langle D \rangle_{\max} = l/2$ and period four for $\langle D \rangle_{\min} = l\sqrt{3}/4$. The geometric mean length of a randomly drawn chord of the triangle is $l\pi\sqrt{3}/12$ and lies between these extrema.

Proof. Let the initial motion of three identical rods on a ring be $y_k = b_k + v_k t$, with t as time and $v_3 > v_2 > v_1$. When rods collide, their identities swap, but their trajectories continue as straight lines. Collisions occur when any of the following are satisfied modulo L:

$$(v_3 - v_2)\, t + b_3 - b_2 = 0, \quad (v_3 - v_1)\, t + b_3 - b_1 = 0, \quad (v_2 - v_1)\, t + b_2 - b_1 = 0$$

Thus the mean collision rate is $\Gamma = 2(v_3 - v_1)/L$. From (1.3) and (1.10), we find $\Gamma = 4U_2/l\sqrt{3}$. The mean distance between impacts is $\langle D \rangle = W/\Gamma$. This yields $\langle D \rangle = \sqrt{3}\, l(4\cos\phi)$, with $\cos\phi = U_2/W$ and U_2 the largest of the initial U_k. Periodic orbits corresponding to the extrema of $\langle D \rangle$ are readily constructed.

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